OPTIMIZATION OF MOTION FOR A TWO-STAGE ROCKET

(OPTIMIZATSIIA DVIZHENIIA DVUSTUPENCHATOI RAKETY)

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Considered is the space motion optimization problem for a two-stage rocket in a homogeneous parallel force field [1 and 2]. The thrust of the power-plants of both stages is considered limited. The results of [3] are utilized in the solution.

1. Formulation of the problem. The equations of motion of a two-stage rocket in a homogeneous parallel force field can be expressed in the form [4 and 5]

$$\mathbf{v}^{+\cdot} = \frac{c^+\beta^+}{M_0^+}\mathbf{e}^+ - g\mathbf{k}, \qquad \mathbf{r}^{+\cdot} = \mathbf{v}^+, \qquad M_0^{+\cdot} = -\beta^+, \qquad \mathbf{e}^+\cdot\mathbf{e}^+ = 1 \tag{1.1}$$

$$\mathbf{v}^{-\cdot} = \frac{c^-\beta^-}{M_0^-}\mathbf{e}^- - g\mathbf{k}, \qquad \mathbf{r}^{-\cdot} = \mathbf{v}^-, \qquad M_0^{-\cdot} = -\beta^-, \qquad \mathbf{e}^-\cdot\mathbf{e}^- = 1 \tag{1.2}$$

Here $\mathbf{r}$ is the radius vector, $\mathbf{v}$ the velocity vector, M_0 the mass of the rocket, c the efflux velocity, $\mathbf{e}$ the thrust unit vector, $\mathbf{k}$ the field forces unit vector, g the gravitational accrleration. The "plus" sign refers to the first stage, while the "minus" refers to the second stage.

Equations (1.1) are valid in the range of the inequality

$$M_0^+(t) \geqslant M_1 \tag{1.3}$$

where M_1 is the rocket mass at the first-stage burnout instant $t = t_1$. Equations (1.2) describe the motion of the second stage of the rocket. It satisfies the inequality

$$M^-(t) \leqslant M_1 - M_c \tag{1.4}$$

where M_c is the "dry" mass of the first stage [6]. The β parameter is regarded as bounded. The intervals of its permissible variations are given by the inequalities

$$\beta_1^{\pm} \leqslant \beta^{\pm} \leqslant \beta_2^{\pm} \tag{1.5}$$

Boundary values of β_1^+ and β_2^+ can differ from β_1^- and β_2^-.

The optimization problem is formulated as follows.

It is required to find among the continuous functions $\mathbf{r}(t)$, $\mathbf{v}(t)$, $M(t)$ and the piecewise continuous parameters $\beta(t)$, $\mathbf{e}(t)$, satisfying the inequalities (1.5), Equations (1.1), (1.2) in the interval $t_0 \leqslant t \leqslant T$ and the relationships

$$\varphi_l\,[\mathbf{r}(t_0), \mathbf{v}(t_0), M(t_0), \mathbf{r}(T), \mathbf{v}(T), M(T), t_0, T] = 0 \quad (l = 1, \ldots, p \leqslant 15) \tag{1.6}$$

at the ends of the interval, such functions which minimize the functional

$$J = J\,[\mathbf{r}(t_0), \mathbf{v}(t_0), M(t_0), t_0, \mathbf{r}(T), \mathbf{v}(T), M(T), T] \tag{1.7}$$

Let us introduce a new variable $M(t)$ by the relations

$$M^+(t) = M_0^+(t), \qquad M^-(t) = M_0^-(t) + M_c \tag{1.8}$$

Then Equations (1.1) and (1.2) become

$$\mathbf{g}_v^+ = \mathbf{v}^{+\cdot} + \frac{c^+\beta^+}{M^+}\mathbf{e}^+ + g\mathbf{k} = 0, \qquad \mathbf{g}_r^+ = \mathbf{r}^{+\cdot} - \mathbf{v}^+ = 0$$
$$g_{\mathcal{M}}^+ = M^{+\cdot} + \beta^+ = 0, \qquad \psi_e^+ = \mathbf{e}^+\cdot\mathbf{e}^+ - 1 = 0 \tag{1.9}$$

$$\mathbf{g}_v^- = \mathbf{v}^{-\cdot} - \frac{c^-\beta^-}{M^- - M_c}\mathbf{e} - g\mathbf{k} = 0, \qquad \mathbf{g}_r^- = \mathbf{r}^{-\cdot} - \mathbf{v}^- = 0$$
$$g_{\mathcal{M}}^- = M^{-\cdot} + \beta^- = 0, \qquad \psi_e^- = \mathbf{e}^-\cdot\mathbf{e}^- - 1 = 0 \tag{1.10}$$

and will be discontinuous in the right-hand side for $t = t_1$, when the relationship $\vartheta = M(t_1) - M_1 = 0$ is fulfilled. Let us construct the auxiliary relationships

$$\psi_\beta^\pm = (\beta^\pm - \beta_1^\mp)(\beta_2^\pm - \beta^\pm) - (u^\pm)^2 = 0 \tag{1.11}$$

Here $u^\pm$ are auxiliary controls. Following this the optimization problem of the motion regimes for a two-stage rocket can be formulated in the following manner.

It is required to find among the continuous functions $\mathbf{r}(t)$, $\mathbf{v}(t)$, $M(t)$ and the piecewise continuous controls $\beta(t)$, $\mathbf{e}(t)$, $u(t)$, satisfying Equations (1.9) to (1.11) in the interval $t_0 \leqslant t \leqslant T$ and the relationships (1.6) at its ends, such functions which minimize the functional (1.7).

In this formulation the problem is a particular case of the problem investigated in [3] in a general problem of optimizing the control processes in the systems described by differential equations with discontinuous right-hand sides.

2. Construction of the equations for the variational problem. Following the rules given in [3], we construct the functions H and φ. They become

$$H^+ = H_\lambda^+ + H_\mu^+ = \boldsymbol{\lambda}_v^+\cdot\left(\frac{c^+\beta^+}{M^+}\mathbf{e}^+ - g\mathbf{k}\right) + \boldsymbol{\lambda}_r^+\mathbf{v}^+ - \lambda_{\mathcal{M}}^+\beta^+ + \mu_e^+(\mathbf{e}^+\cdot\mathbf{e}^+ - 1) +$$
$$+ \mu_\beta^+[(\beta^+ - \beta_1^+)(\beta_2^+ - \beta^+) - (u^+)^2] \tag{2.1}$$

$$H^- = H_\lambda^- + H_\mu^- = \boldsymbol{\lambda}_v^-\cdot\left(\frac{c^-\beta^-}{M^- - M_c}\mathbf{e}^- - g\mathbf{k}\right) + \boldsymbol{\lambda}_r^-\cdot\mathbf{v}^- - \lambda_{\mathcal{M}}^-\beta^- + \mu_e^-(\mathbf{e}^-\cdot\mathbf{e}^- - 1) +$$
$$+ \mu_\beta^-[(\beta^- - \beta_1^-)(\beta_2^- - \beta^-) - (u^-)^2] \tag{2.2}$$

$$\varphi = J + \sum_{l=1}^{p} \rho_l\varphi_l \tag{2.3}$$

By utilizing Formulas (2.13) and (2.14) of the paper [3], we find Equations (2.4)

$$\boldsymbol{\lambda}_v^+ + \boldsymbol{\lambda}_r^+ = 0, \qquad \boldsymbol{\lambda}_r^{+\cdot} = 0, \qquad \lambda_{\mathcal{M}}^{+\cdot} - \frac{c^+\beta^+}{(M^+)^2}\boldsymbol{\lambda}_v^+\cdot\mathbf{e}^+ = 0, \qquad \frac{c^+\beta^+}{M^+}\boldsymbol{\lambda}_v^+ + 2\mu_e^+\mathbf{e}^+ = 0$$

$$\frac{c^+}{M^+}\boldsymbol{\lambda}_v^+\cdot\mathbf{e}^+ - \lambda_{\mathcal{M}}^+ - \mu_\beta^+(2\beta^+ - \beta_1^+ - \beta_2^+) = 0, \qquad \mu_\beta^+u^+ = 0 \tag{2.5}$$

$$\boldsymbol{\lambda}_v^{-\cdot} + \boldsymbol{\lambda}_r^- = 0, \qquad \boldsymbol{\lambda}_r^{-\cdot} = 0, \qquad \lambda_{\mathcal{M}}^{-\cdot} - \frac{c^-\beta^-}{(M^- - M_c)^2}\boldsymbol{\lambda}_v^-\cdot\mathbf{e}^- = 0 \tag{2.6}$$
$$\frac{c^-\beta^-}{M^- - M_c}\boldsymbol{\lambda}_v^- + 2\mu_e^-\mathbf{e}^- = 0$$

$$\frac{c^-}{M^- - M_c}\boldsymbol{\lambda}_v^-\cdot\mathbf{e}^- - \lambda_{\mathcal{M}}^- - \mu_\beta^-(2\beta^- - \beta_1^- - \beta_2^-) = 0, \qquad \mu_\beta^-u^- = 0 \tag{2.7}$$

With the aid of the relations (2.16) and (2.22) of [3] we construct the end conditions

$$\lambda_r(t_0) = \frac{\partial\varphi}{\partial \mathbf{r}(t_0)}, \qquad \lambda_v(t_0) = \frac{\partial\varphi}{\partial \mathbf{v}(t_0)}, \qquad \lambda_M(t_0) = \frac{\partial\varphi}{\partial M(t_0)}$$

$$\lambda_r(T) = -\frac{\partial\varphi}{\partial \mathbf{r}(T)}, \qquad \lambda_v(T) = -\frac{\partial\varphi}{\partial \mathbf{v}(T)}, \qquad \lambda_M(T) = -\frac{\partial\varphi}{\partial M(T)} \tag{2.8}$$

and the functions

$$\frac{\partial\varphi}{\partial t_0} = -(H)_{t_0}, \qquad \frac{\partial\varphi}{\partial T} = (H)_T \tag{2.9}$$

Finally, Formulas (2.17) and (2.33) of [3] lead to the following Erdmann-Weierstrass conditions:

$$\lambda_r^+(t_1) - \lambda_r^-(t_1) = 0, \qquad \lambda_v^+(t_1) - \lambda_v^-(t_1) = 0, \qquad \lambda_M^+(t_1) - \lambda_M^-(t_1) + \nu = 0, \tag{2.10}$$

$$(H^+)_{t_1} - (H^-)_{t_1} = 0 \tag{2.11}$$

We also note that the problem equations do not contain time explicitly, and that there exists the first integral

$$H = h = \text{const} \tag{2.12}$$

In the optimum regime there must also be fulfilled the inequality

$$(\eta\beta)^\pm \geqslant (\eta^*\beta^*)^\pm \tag{2.13}$$

in which

$$\eta^+ = \frac{c^+}{M^+}\lambda_v^+\cdot \mathbf{e}^+ - \lambda_M^+, \qquad \eta^{*+} = \frac{c^+}{M^+}\lambda_v^+\cdot \mathbf{e}^{*+} - \lambda_M^+$$

$$\eta^- = \frac{c^-}{M^- - M_c}\lambda_v^-\cdot \mathbf{e}^- - \lambda_M^-, \qquad \eta^{*-} = \frac{c^-}{M^- - M_c}\lambda_v^-\cdot \mathbf{e}^{*-} - \lambda_M^- \tag{2.14}$$

and $\mathbf{e}^\pm$, $\beta^\pm$ correspond to the optimum regime, while $\mathbf{e}^{*\pm}$ and $\beta^{*\pm}$ are any permissible functions. The inequality (2.13) was derived from the relation (3.4) of [3] after substitution in it of the function H represented by Equalities (2.1) and (2.2).

3. Construction of the optimum regime. Let us consider Formulas (2.5) and (2.7). They show that the vectors λ_v and $\mathbf{e}$ are parallel and they lead to the conclusion that in the optimum regime the following systems of dependences can be fulfilled:

$$1.\quad \mu_\beta = 0,\quad u \neq 0; \qquad 2.\quad \mu_\beta \neq 0,\quad u = 0; \qquad 3.\quad \mu_\beta = u = 0$$

The second equalities of (2.5) and (2.7) permit to verify that for the case $\mu_\beta = 0$ the corresponding $\eta = 0$, and for $\mu_\beta \neq 0$ the inequality $\eta \neq 0$ is fulfilled.

Let us refer to the inequality (2.13). Noting that the function $\eta^* = \eta$ is permissible we substitue it into the inequality. This results in the relationship $\eta\beta \geqslant \eta\beta^*$.

Consequently, for $\eta \neq 0$, we get

$$\beta^\pm = \begin{cases} \beta_1^\pm & \text{for } \eta^\pm < 0 \\ \beta_2^\pm & \text{for } \eta^\pm > 0 \end{cases} \tag{3.1}$$

The following relationships correspond to two other cases:

$$1.\ \beta_1^\pm < \beta^\pm < \beta_2^\pm;\ 2.\ \eta^\pm = 0;\ 3.\ \beta^\pm = \beta_{1,2}^{\ \pm},\ \eta^\pm = 0$$

For $\eta \neq \eta^*$, $\beta = \beta^*$ we find $\eta \geqslant \eta^*$ or,

$$\frac{c^+}{M^+}\lambda_v^+\cdot \mathbf{e}^+ \geqslant \frac{c^+}{M^+}\lambda_v^+\cdot \mathbf{e}^{*+}, \qquad \frac{c^-}{M^- - M_c}\lambda_v^-\cdot \mathbf{e}^- \geqslant \frac{c^-}{M^- - M_c}\lambda_v^-\cdot \mathbf{e}^{*-} \tag{3.2}$$

The analysis of these inequalities shows that in the optimum regime the vectors λ_v and $\mathbf{e}$ are codirectional so that

$$\lambda_v^{\pm} = \lambda_v^{\pm} \mathbf{e}^{\pm} \qquad (3.3)$$

where $\lambda_v^{\pm}$ are the lengths of the vectors $\lambda_v^{\pm}$.

With the aid of the conditions (2.10) and (2.11) we find that the vector multipliers $\lambda_r(t)$ and $\lambda_v(t)$ and the function H are continuous in the entire interval $t_0 \leqslant t \leqslant T$. The multiplier $\lambda_{\mathcal{M}}(t)$ and the function η can have the first kind of discontinuity at the point $t = t_1$.

Substituting into the equality (2.11) Expressions (2.1) and (2.2) there results the relationship

$$\eta^+(t_1)\,\beta^+(t_1) = \eta^-(t_1)\,\beta^-(t_1) \qquad (3.4)$$

Since $\beta^{\pm} > 0$, the values of $\eta^+(t_1)$ and $\eta^-(t_1)$ are of the same sign. At the point $t = t_1$ there is no change of regimes and if $\beta^+(t_1) = \beta_2^+$, then $\beta^-(t_1) = \beta_2^-$ while for $\beta^+(t_1) = \beta_1^+$ we will get $\beta^-(t_1) = \beta_1^-$.

Let us examine the case of given initial and final positions of the rocket by the relations

$$\mathbf{r}(0) = \mathbf{r}^\circ, \qquad \mathbf{v}(0) = \mathbf{v}^\circ, \qquad M(0) = M^\circ, \qquad t_0 = (0) \qquad (3.5)$$

$$\mathbf{r}(T) = \mathbf{r}^T, \qquad \mathbf{v}(T) = \mathbf{v}^T \qquad (3.6)$$

The J functional is

$$J = J\,[M(T),\, T] \qquad (3.7)$$

The φ function is represented by the equality

$$\varphi = J + \rho_r^\circ(\mathbf{r}(0) - \mathbf{r}^\circ) + \rho_v^\circ(\mathbf{v}(0) - \mathbf{v}^\circ) + \rho_{\mathcal{M}}(M(0) - M^\circ) + \\ + \rho_{t_0} t_0 + \rho_r(\mathbf{r}(T) - \mathbf{r}^T) + \rho_v(\mathbf{v}(T) - \mathbf{v}^T) \qquad (3.8)$$

The first group of relationships in (2.8) and the first function in (2.9) derive Formulas

$$\lambda_r(0) = \rho_r^\circ, \quad \lambda_v(0) = \rho_v^\circ, \quad \lambda_{\mathcal{M}}(0) = \rho_{\mathcal{M}}^\circ, \quad (H)_{t_0} = -\rho_{t_0} \qquad (3.9)$$

so that instead of seeking the multipliers $\rho_r^\circ, \rho_v^\circ, \rho_{\mathcal{M}}^\circ, \rho_{t_0}$ we can compute the initial values of the corresponding functions.

On the basis of the second group of conditions in (2.8) and the second equality in (2.9) we get

$$\lambda_r(T) = -\rho_r, \qquad \lambda_v(T) = -\rho_v$$

Integrating Equations (2.4) and (2.6) with these relationships taken into account, we will obtain

$$\lambda_r^{\pm}(t) = -\rho_r, \qquad \lambda_v^{\pm}(t) = \rho_r(t - T) - \rho_v$$

Let us construct now the derivative $\eta^\cdot$ and find

$$\eta^{+\cdot} = -\frac{c^+}{M^+}\lambda_r^+\cdot \mathbf{e}^+, \qquad \eta^{-\cdot} = -\frac{c^-}{M^- - M_c}\lambda_r^-\cdot \mathbf{e}^- \qquad (3.10)$$

The substitution of $\lambda_r^{\pm}(t)$ into these expressions yields

$$\eta^{+\cdot} = -\frac{c^+}{M^+\lambda_v^+}[\rho_r\cdot\rho_v - \rho_r^2(t - T)], \qquad \eta^{-\cdot} = -\frac{c^-}{(M^- - M_c)\lambda_v^-}[\rho_r\cdot\rho_v - \rho_r^2(t - T)] \qquad (3.11)$$

Here ρ_r is the length of the vector $\boldsymbol{\rho}_r$, and

$$\lambda_v = \sqrt{\rho_v^2 - 2\rho_v\cdot\rho_r(t - T) + \rho_r^2(t - T)^2}$$

Formulas (3.11) show that for $\rho_r \neq 0$ the derivative $\eta^\cdot$ cannot have more than one zero in the interval $t_0 \leqslant t \leqslant T$. If $\eta^\cdot = 0$ and $\eta = 0$, then at the neighboring points $\eta \neq 0$. Therefore, the function η can become zero at a finite number of points in the interval $t_0 \leqslant t \leqslant T$; at all other points it is nonzero.

Since the left and right regions of the function η at a point $t = t_1$ have the same signs and its derivative has no more than a single zero, the function η itself can have no more than two zeros. Thus, in specifying the end values by the equalities (3.5) and (3.6), the control parameter in the optimum regime can assume only its boundary values $\beta^\pm = \beta_1{}^\pm$, or $\beta^\pm = \beta_2{}^\pm$. The change of the regime, i.e. the change of the control parameter from $\beta = \beta_1$ to $\beta = \beta_2$ or conversely can occur only when $\eta = 0$. There can be no more than two corresponding instants of time. In addition, there is a point of discontinuity $t = t_1$ in the right-hand parts of the equations of motion where the control parameter changes from $\beta_1{}^+$ to $\beta_1{}^-$ or from $\beta_2{}^+$ to $\beta_2{}^-$.

4. Vertical motions of the rocket. In the absence of drag the equations for the vertical motion of the rocket are

$$w^{+\cdot} - \frac{c^+\beta^+}{M^+} + g = 0, \quad w^{-\cdot} - \frac{c^-\beta^-}{M^- - M_c} + g = 0, \quad z^{\pm\cdot} - w^\pm = 0, \qquad M^{\pm\cdot} + \beta^\pm = 0 \tag{4.1}$$

Here z is the vertical coordinate, w the vertical velocity. The end conditions will be specified by

$$z(0) = z^\circ, \quad w(0) = w^\circ, \quad M(0) = M^\circ, \quad \varphi_z = z(T) - z^T = 0 \tag{4.2}$$

We will consider the problem of fuel consumption minimization. Then

$$J = -M(T) \tag{4.3}$$

Utilizing the above derived functions we obtain Equations

$$\lambda_w{}^{\pm\cdot} - \lambda_z{}^\pm = 0, \quad \lambda_z{}^{\pm\cdot} = 0, \quad \lambda_M{}^{+\cdot} - \frac{c^+\beta^+}{(M^+)}\lambda_w{}^+ = 0, \quad \lambda_M{}^- - \frac{c^-\beta^-}{(M^- - M_c)^2}\lambda_w{}^- = 0$$

$$\frac{c^+}{M^+}\lambda_w{}^+ - \lambda_M{}^+ - \mu_\beta{}^+(2\beta^+ - \beta_1{}^+ - \beta_2{}^+) = 0 \tag{4.4}$$

$$\frac{c^-}{M^- - M_c}\lambda_w{}^- - \lambda_M{}^- - \mu_\beta{}^-(2\beta^- - \beta_1{}^- - \beta_2{}^-) = 0, \qquad \mu_\beta{}^\pm u^\pm = 0$$

and the end conditions for the λ multipliers

$$\lambda_z(T) = -\rho_z, \qquad \lambda_w(T) = 0, \qquad \lambda_M(T) = 1 \tag{4.5}$$

In addition, there will exist the equality $(H)_T = 0$. For the $\eta^\pm$ functions we get

$$\eta^+ = \frac{c^+}{M^+}\lambda_z{}^+ - \lambda_M{}^+, \qquad \eta^- = \frac{c^-}{M^- - M_c}\lambda_w{}^- - \lambda_M{}^- \tag{4.6}$$

and the Weierstrass inequality will be of the form $(\eta\beta)^\pm \geqslant (\eta\beta^*)^\pm$.

The derivatives $\eta^\cdot$ are

$$\eta^+ = -\frac{c^+}{M^+}\lambda_z{}^+, \qquad \eta_z{}^{-\cdot} = -\frac{c^-}{M^- - M_c}\lambda_z{}^- \tag{4.7}$$

The Erdmann-Weierstrass conditions for the point of discontinuity in the right-hand part of the equations of motion are expressed as follows:

$$\begin{gathered}\lambda_z{}^+(t_1) - \lambda_z{}^-(t_1) = 0, \qquad \lambda_w{}^+(t_1) - \lambda_w{}^-(t_1) = 0 \\ \lambda_M{}^+(t_1) - \lambda_M{}^-(t_1) + \nu = 0, \qquad (H^+)_{t_1} - (H^-)_{t_1} = 0\end{gathered} \tag{4.8}$$

The corresponding conditions for the points of discontinuity of the control parameters are equivalent to the requirement of continuity for the multipliers $\lambda_z(t)$, $\lambda_w(t)$, $\lambda_M(t)$ and the function H.

Integrating Equations (4.4) under the conditions (4.5), there results

$$\lambda_z = -\rho_z, \qquad \lambda_w = -\rho_z(t - T) \tag{4.9}$$

Substituting λ_z into the formula for $\eta^\cdot$, we see that this derivative does not change sign in the interval $t_0 \leqslant t \leqslant T$. Consequently, the η function cannot have more than one zero in this interval. At all other points we will have $\eta \neq 0$. For $t = t^*$, $(\eta(t^*) = 0)$ the change of regimes occurs.

Let us examine the right end of the optimum trajectory. Substituting $t = T$ into the second formula in (4.6) and utilizing the equalities (4.4) we get

$$\eta^\pm(T) = -\lambda_M{}^\pm(T) = -1 < 0 \tag{4.10}$$

Therefore, the optimum regime ends when

$$\beta^{\pm}(T) = \beta_1^{\pm} \tag{4.11}$$

The second characteristic of the right end of the trajectory is found with the aid of the equality $H = 0$. Substituting $t = T$ into it we get

$$(H^{\pm})_T = \lambda_z^{\pm}(T)\, w^{\pm}(T) - \beta^{\pm}(T) = 0, \qquad \text{or} \qquad w^{\pm}(T) = -\frac{\beta_1^{\pm}}{\beta_z}$$

For $\beta_1^{\pm} = 0$ the above derived results simplify and yield the relationships $\beta^{\pm}(T) = 0$, $w^{\pm}(T) = 0$.

For two-stage rockets of this type there can be the following optimum regimes:

$$\beta^{(1)} = \begin{cases} \beta_2^{+} & (0 \leqslant t \leqslant t^*) \\ 0 & (t^* \leqslant t \leqslant T) \end{cases} \tag{4.12}$$

$$\beta^{(2)} = \begin{cases} \beta_2^{+} & (0 \leqslant t \leqslant t_1) \\ \beta_2^{-} & (t_1 \leqslant t \leqslant t^*) \\ 0 & (t^* \leqslant t \leqslant T) \end{cases} \tag{4.13}$$

The first one corresponds to the case when the delivery of the rocket to the given height $z(T) = z^T$ with velocity $w(T) = 0$ can be achieved with the aid of the first stage fuel supply; the second uses the powerplant of the second stage.

The optimum regimes can be constructed without determining the Lagrange multipiers. This requires finding the solution of the equations of motion for the values of β given by the relations (4.12) or (4.13) such that for $t = T$ the equalities $z(T) = z^T$ and $w(T) = 0$ would be satisfied.

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